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Environment-induced nonclassical behaviour

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Abstract. We analyze the transient nonclassical behaviour of a single-mode field whose interaction with an environment is governed by the quantum optical master equation. Our analytic method makes use of the generalized characteristic function of the field state. First, we find a time at which all squeezing effects disappear by decoherence regardless of the initial state of the mode. In the case of an input even coherent state, an unusual modification of higher-order squeezing at low values of thermal mean occupancy transferred to the field is found and discussed. For the same initial state, we also perform a comprehensive analysis of the mixing process during the interaction with the reservoir. We prove that a maximum in the evolution of the 2-entropy of the attenuated mode exists on condition that its initial mean photon number exceeds the mean occupancy of the reservoir. This transient mixing enhancement can be considered as a quantum effect of the initial state on the mode damping.

PACS. 42.50.Dv Nonclassical field states; squeezed, antibunched, and sub-Poissonian states; operational definitions of the phase of the field; phase measurements – 03.65.Bz Foundations, theory of measurement, miscellaneous theories (including Aharonov-Bohm effect, Bell inequalities, Berry's phase)

1 Introduction

Interaction of a quantum system, such as a single-mode radiation field, with an environment has been intensively studied in connection with the emergency of classicality for the system [1–3]. In reference [1], the master equation of the quantum Brownian motion in the high-temperature limit has been used to get a predictability sieve for the effectiveness of quantum decoherence. In references [2,3], effects of the environment on a harmonic oscillator have been investigated in the framework of the Lindblad equation formalism [4]. Special attention is paid to the case of an interaction which is linear in position and momentum [5]. The works [1–3] have exploited the production of reduced linear entropy as an instrument for determining maximal states of the system. Examination of the von Neumann entropy production deduced from the master equation of quantum Brownian motion [6] reveals that information about the system diminishes considerably at short times. Simultaneously, dramatic changes of nonclassical properties of the field mode could happen. These properties generally soften owing to the dissipative interaction [7–9]. However, recent works [10–12] report that, in some cases, environment enhances nonclassical features. Specifically, a single-mode field in a superposition of coherent states weakly coupled to a heat bath at zero [10] or very low temperature [11] has been studied. It has been found that fourth-order squeezing could be transiently created due to the interaction with the reservoir described by the quantum optical master equation [13]. Note that, according to references [2–5], this master equation is precisely of the type introduced by Lindblad.

Use will be made in the present paper of the quantum optical master equation in order to elucidate two problems:

(i) occurrence of higher-order squeezing for a nonclassical single-mode radiation field coupled to a low-temperature heat bath. As an illustration, the case of an initial *even coherent state* (ECS) [14] is explicitly treated;
(ii) evolution of the mixing process undergone by an input ECS.

The paper is organized as follows. In Section 2 we first discuss the solution of the master equation in terms of the characteristic function (CF) of the damped mode. Then, we give general analytic formulae describing squeezing to any even order N. They are applied in Section 3 to the interesting case of an ECS. We also point out the loss of nonclassicality by means of the P-representation of the density operator. The 2-entropy of a damped ECS is evaluated in Section 4 making use of its CF. Clearly, the 2-entropy turned out to be an efficient tool to analyze the evolution of the mixing produced by the environment. We distinguish a classical behaviour of this mixing from a nonclassical one by comparing the initial mean photon number of the field with the mean occupancy of the reservoir.

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2 Evolution of higher-order squeezing

We denote by a and $a^\dagger$ the amplitude operators of the field mode. The quantum optical master equation in the interaction picture is [13]

$$\frac{\partial\rho}{\partial t} = \frac{\gamma}{2}(\bar{n}_{\rm R}+1)(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) + \frac{\gamma}{2}\bar{n}_{\rm R}(2a^\dagger\rho a - aa^\dagger\rho - \rho aa^\dagger). \quad (2.1)$$

In equation (2.1), ρ is the reduced density operator of the field, γ is the coupling constant between field and bath, and $\bar{n}_{\rm R}$ stands for the mean occupancy of the reservoir.

The usefulness of the normally ordered generalized characteristic function (GCF) recently introduced in our paper [12] as

$$\chi_{\rm N}(\lambda,\lambda') := \langle \exp(\lambda a^\dagger)\exp(\lambda' a)\rangle \quad (2.2)$$

is proved once more when applying it to equation (2.1). By use of well-known methods [7,13], we convert the master equation (2.1) into a first-order partial differential equation for the GCF (2.2):

$$\frac{\partial\chi_{\rm N}}{\partial t} = \gamma\bar{n}_{\rm R}\lambda\lambda'\chi_{\rm N} - (\frac{\gamma}{2}-{\rm i}\omega)\lambda\frac{\partial\chi_{\rm N}}{\partial\lambda} - (\frac{\gamma}{2}+{\rm i}\omega)\lambda'\frac{\partial\chi_{\rm N}}{\partial\lambda'}\cdot \quad (2.3)$$

The solution of this equation found by the characteristic-curve method [15] depends on its initial form $\chi_{\rm N}(\lambda,\lambda',0)$ as

$$\chi_{\rm N}(\lambda,\lambda',t) = \chi_{\rm N}(\lambda{\rm e}^{-(\gamma/2-{\rm i}\omega)t}, \lambda'{\rm e}^{-(\gamma/2+{\rm i}\omega)t},0) \times \exp\left[\bar{n}_{\rm R}(1-{\rm e}^{-\gamma t})\lambda\lambda'\right]. \quad (2.4)$$

Note that the GCF (2.2) is picture independent. When $\lambda' = -\lambda^*$, equation (2.2) gives the usual normally ordered CF $\chi_N(\lambda)$ introduced by Glauber [16]. By employing this function, one readily finds the expectation values

$$\langle (a^\dagger)^l a^m\rangle = (-1)^m\left[\frac{\partial^{l+m}}{\partial\lambda^l\partial\lambda^{*m}}\chi_{\rm N}(\lambda)\right]_{\lambda=0}, \quad (2.5)$$

which are necessary when one has to examine the statistical properties of the field state. For instance, the mean photon number ($l=m=1$ in Eq. (2.5)) is obtained from the normally ordered CF (2.4) as

$$\bar{n}(t) = \bar{n}(0){\rm e}^{-\gamma t} + \bar{n}_{\rm T}(t), \quad (2.6)$$

where

$$\bar{n}_{\rm T}(t) := \bar{n}_{\rm R}[1-\exp(-\gamma t)] \quad (2.7)$$

is the thermal mean occupancy in the field mode at time t. Note also that, if existing as a well-behaved function, the Fourier transform of the normally ordered CF is the Glauber-Sudarshan P-representation [17]

$$P(\beta) = \frac{1}{\pi}\int {\rm d}^2\lambda\ \exp(\beta\lambda^* - \beta^*\lambda)\chi_{\rm N}(\lambda). \quad (2.8)$$

The density operator is fully determined by the symmetrically ordered CF,

$$\chi(\lambda) = \exp(-|\lambda|^2/2)\chi_{\rm N}(\lambda), \quad (2.9)$$

via the Weyl expansion [18]

$$\rho = \frac{1}{\pi}\int {\rm d}^2\lambda\ \chi(\lambda)D(-\lambda). \quad (2.10)$$

In equation (2.10), $D(\beta) = \exp(\beta a^\dagger - \beta^* a)$ is a Weyl displacement operator.

Now, the significance of the factorization (2.4) is quite transparent: it describes the superposition of the attenuated field with a thermal one whose time-dependent mean occupancy is $\bar{n}_{\rm T}(t)$ (Eq. (2.7)). Therefore, the decay of the field mode ruled by the quantum optical master equation is a *thermalization process*, as studied generically in our paper [12]. We can now use the generating-function method presented in reference [12] in order to obtain the time-dependent higher-order moments of the quadrature operators as functions of the similar ones at the initial moment $t=0$. Due to the oscillatory factors in the CF (2.4), such expectation values are also rapidly oscillating functions. In what follows, we give only formulae with the oscillatory factors removed. Application of the steps described in reference [12] to the factorization (2.4) yields the following formulae, valid for an even order N and $j=1,2$:

$$\langle(\Delta X_j)^N\rangle_t = N!\sum_{m=0}^{[N/2]}\frac{[2\bar{n}_{\rm T}(t)+1]^m\exp(-\frac{N-2m}{2}\gamma t)}{2^{3m}m!(N-2m)!} \times \langle :(\Delta X_j)^{N-2m}:\rangle_0, \quad (2.11a)$$

$$\langle(\Delta X_j)^N\rangle_t = N!\sum_{m=0}^{[N/2]}\frac{[2\bar{n}_{\rm T}(t)+1-{\rm e}^{-\gamma t}]^m\exp(-\frac{N-2m}{2}\gamma t)}{2^{3m}m!(N-2m)!} \times \langle(\Delta X_j)^{N-2m}\rangle_0, \quad (2.11b)$$

$$\langle :(\Delta X_j)^N:\rangle_t = N!\sum_{m=0}^{[N/2]}\frac{[\bar{n}_{\rm T}(t)]^m\exp(-\frac{N-2m}{2}\gamma t)}{2^{2m}m!(N-2m)!} \times \langle :(\Delta X_j)^{N-2m}:\rangle_0, \quad (2.11c)$$

$$\langle :(\Delta X_j)^N:\rangle_t = N!\sum_{m=0}^{[N/2]}\frac{[2\bar{n}_{\rm T}(t)-{\rm e}^{-\gamma t}]^m\exp(-\frac{N-2m}{2}\gamma t)}{2^{3m}m!(N-2m)!} \times \langle(\Delta X_j)^{N-2m}\rangle_0. \quad (2.11d)$$

Here $X_1 := (a+a^\dagger)/2$ and $X_2 := (a-a^\dagger)/(2{\rm i})$ are the quadrature operators, $\Delta X_j := X_j - \langle X_j\rangle$, and the symbol $:\,:$ denotes the normal-ordering operation. We briefly examine the structure of these equations in connection with the concept of higher-order squeezing [19].

2.1 Intrinsic higher-order squeezing

As $\langle(\Delta X_j)^N\rangle$ is always positive for even N, from equation (2.11d) we learn that the condition for intrinsic squeezing, $\langle : (\Delta X_j)^N :\rangle_t < 0$, cannot be fulfilled if $2\bar{n}_{\rm T}(t) - {\rm e}^{-\gamma t}$ is positive. Accordingly, the time

$$t_{\rm s} := \frac{1}{\gamma}\ln(1+\frac{1}{2\bar{n}_{\rm R}}) \tag{2.12}$$

is the upper limit for *existence of intrinsic squeezing to any order for arbitrary input states.* Equation (2.11c) displays a remarkable effect: even if the input field state were not intrinsically squeezed to order N, interaction with the heat bath might in principle lead to the appearance of Nth-order intrinsic squeezing due to the contributions of lower-order moments at $t = 0$ in the r.h.s. of equation (2.11c).

2.2 Ordinary higher-order squeezing

Recall that the negative values of the coefficient

$$q_{\rm N} = \frac{\langle(\Delta X_1)^N\rangle}{2^{-N}(N-1)!!} - 1, \tag{2.13}$$

are a manifestation of Nth-order squeezing, as introduced by Hong and Mandel [19]. Nevertheless, the condition $t < t_{\rm s}$ holds also in the case of ordinary squeezing as a consequence of the direct relation between ordinary and normally ordered moments (McCoy's theorem). From the general equations (2.11a, 2.11b), we learn that squeezing to order N at time t is determined by the presence of the same property to orders $N' < N$ at time $t = 0$. This may result in an enhancement of transient higher-order squeezing with respect to the corresponding initial one. However, in order to draw such a conclusion, one needs to examine the squeezing properties of the input state. We have chosen to discuss such issues for an ECS, that is to say a superposition of states which has been much used recently to study decoherence [10,20,21].

3 Input even coherent state

An ECS is a definite superposition of two coherent states,

$$|\alpha\rangle_{\rm e} = \mathcal{N}(|\alpha\rangle + |-\alpha\rangle), \tag{3.1}$$

with the normalization factor

$$\mathcal{N} = [2(1+\exp(-2|\alpha|^2))]^{-1/2}.$$

When inserted into equation (2.8), the normally ordered CF for an ECS,

$$\chi_{\rm N}(\lambda,0) = \mathcal{N}^2\left[{\rm e}^{\lambda\alpha^*-\lambda^*\alpha} + {\rm e}^{-\lambda\alpha^*+\lambda^*\alpha} + {\rm e}^{-2|\alpha|^2}({\rm e}^{\lambda\alpha^*+\lambda^*\alpha} + {\rm e}^{-\lambda\alpha^*-\lambda^*\alpha})\right], \tag{3.2}$$

leads to a P-representation which is not a well-behaved function. Therefore, the ECS is a nonclassical state.

In reference [12], we have calculated the normally ordered moments of the quadrature operator X_2 for an ECS as

$$\langle : (\Delta X_2)^N :\rangle_{\rm e} = \frac{(-1)^{N/2}}{2}\left[(1-\tanh(|\alpha|^2))(\Re e(\alpha))^N + (1+\tanh(|\alpha|^2))({\rm i}\Im m(\alpha))^N\right], \tag{3.3}$$

Note also that the mean photon number in the input ECS, equation (3.1), is

$$\bar{n}(0) = |\alpha|^2\tanh(|\alpha|^2). \tag{3.4}$$

3.1 P-representation

We proceed now with the examination of the ECS mixed with thermal noise according to the master equation (2.1). In the interaction picture, the P-representation is the Fourier transform of the nonoscillating normally ordered CF:

$$\chi_{\rm NI}(\lambda,t) := \chi_{\rm N}(\lambda{\rm e}^{-\gamma/2t}, 0)\exp\left(-\bar{n}_{\rm T}(t)|\lambda|^2\right). \tag{3.5}$$

The occurrence of the factor $\exp\left(-\bar{n}_{\rm T}(t)|\lambda|^2\right)$ under the integral (2.8) ensures its existence, but it is still questionable if $\mathcal{P}(\beta,t)$ is positive. When using equation (3.2) in equation (2.8) we are simply left to calculate a sum of Gaussian integrals (see, for instance, Ref. [22], Appendix A). We finally get

$$\mathcal{P}(\beta,t) = \frac{2|\mathcal{N}|^2}{\bar{n}_{\rm T}(t)}\exp\left(-\frac{|\beta|^2{\rm e}^{\gamma t}+|\alpha|^2}{2\bar{n}_{\rm R}\sinh(\gamma t/2)}\right) \times\left\{\cosh\left[\frac{\Re e(\alpha\beta^*)}{\bar{n}_{\rm R}\sinh(\gamma t/2)}\right] + \exp\left[-\frac{|\alpha|^2}{\bar{n}_{\rm R}\sinh(\gamma t/2)} \times[\bar{n}_{\rm R}{\rm e}^{\gamma t/2} - (\bar{n}_{\rm R}+1){\rm e}^{-\gamma t/2}]\right]\cos\left[\frac{\Im m(\alpha\beta^*)}{\bar{n}_{\rm R}\sinh(\gamma t/2)}\right]\right\}. \tag{3.6}$$

If the amplitude of the oscillating function in equation (3.6) is less than unity, the P-representation is positive for any β. This implies a *sufficient* condition for the existence of the P-representation as a well-behaved function:

$$\frac{\bar{n}_{\rm R}}{\bar{n}_{\rm R}+1} > {\rm e}^{-\gamma t}. \tag{3.7}$$

At the time

$$t_{\rm c} := \frac{1}{\gamma}\ln\left(1+\frac{1}{\bar{n}_{\rm R}}\right), \tag{3.8}$$

all the nonclassical properties of the initial ECS dissapear due to the interaction with the heat bath. Obviously, $t_{\rm c} > t_{\rm s}$.

To conclude, we point out that only at times $t < t_{\rm c}$ the ECS coupled to a thermal reservoir is still nonclassical when its P-representation is negative. Nonclassical properties such as higher-order squeezing could survive at $t < t_{\rm s} < t_{\rm c}$, as indicated by our equations (2.11), valid for an arbitrary input state.

3.2 Transient higher-order squeezing

In references [10,11], it was shown that the ECS presents second- and fourth-order squeezing in the quadrature X_2. Second-order squeezing occurs for any α, the maximum degree of squeezing being $q_2 = -0.55$ for $\alpha \approx 0.8$. Fourth-order squeezing is manifest only for $\alpha < 1.23$. When coupled to a thermal bath at zero [10] or low temperature [11], it appears that fourth-order squeezing is also manifest for $\alpha > 1.23$.

In the following, we show that squeezing properties to any order for a damped ECS are described by simple and versatile analytic formulae. When inserting the normally ordered moments for an ECS (3.3) into equations (2.11a, 2.11c), we get the time development of higher-order squeezing:

$$\langle(\Delta X_2)^N\rangle_t = \frac{(-1)^{N/2}[\bar{n}_{\mathrm{T}}(t)+1/2]^{N/2}}{2^{N+1}} \times \left[(1-\tanh(|\alpha|^2))H_{\mathrm{N}}\left(\frac{\Re e(\alpha)\mathrm{e}^{-\gamma t/2}}{\sqrt{\bar{n}_{\mathrm{T}}(t)+1/2}}\right) +(1+\tanh(|\alpha|^2))H_{\mathrm{N}}\left(\frac{\mathrm{i}\Im m(\alpha)\mathrm{e}^{-\gamma t/2}}{\sqrt{\bar{n}_{\mathrm{T}}(t)+1/2}}\right)\right], \quad (3.9)$$

and

$$\langle:(\Delta X_2)^N:\rangle_t = \frac{(-1)^{N/2}[\bar{n}_{\mathrm{T}}(t)]^{N/2}}{2^{N+1}} \times \left[(1-\tanh(|\alpha|^2))H_{\mathrm{N}}\left(\frac{\Re e(\alpha)\mathrm{e}^{-\gamma t/2}}{\sqrt{\bar{n}_{\mathrm{T}}(t)}}\right) +(1+\tanh(|\alpha|^2))H_{\mathrm{N}}\left(\frac{\mathrm{i}\Im m(\alpha)\mathrm{e}^{-\gamma t/2}}{\sqrt{\bar{n}_{\mathrm{T}}(t)}}\right)\right]\cdot \quad (3.10)$$

In equations (3.9, 3.10), $H_{\mathrm{N}}(x)$ is a Hermite polynomial of degree N. According to equation (3.3), the ECS with real α (the case we are interested in) does not possess intrinsic squeezing for even values of $N/2$. Plots of the degree of squeezing (2.13) and of the degree of intrinsic squeezing,

$$q_{\mathrm{N}}^{(\mathrm{i})} := \frac{\langle:(\Delta X_2)^N:\rangle}{2^{-N}(N-1)!!}, \quad (3.11)$$

for the quadrature X_2 are shown in Figures 1 and 2 for an ECS coupled to a thermal bath with the mean occupancy $\bar{n}_{\mathrm{R}} = 0.1$ ($\gamma t_{\mathrm{s}} = 1.791$). In Figure 1 dependences of $q_8^{(\mathrm{i})}$ and q_8 of the coherent amplitude α are given for some values of γt. The same parameters are plotted in Figure 2 *versus* γt for several values of α. We see the occurrence at $t > 0$ of eighth-order intrinsic squeezing which is absent at $t = 0$. Similarly, ordinary eighth-order squeezing is generated at $t > 0$ for values of α for which it is not manifest at $t = 0$. For example, we can see in Figure 1b that the parameter q_8 for a free ECS is positive for $\alpha = 0.9$. However, Figure 2b clearly displays eighth-order squeezing when the ECS with $\alpha = 0.9$ is coupled to the bath.

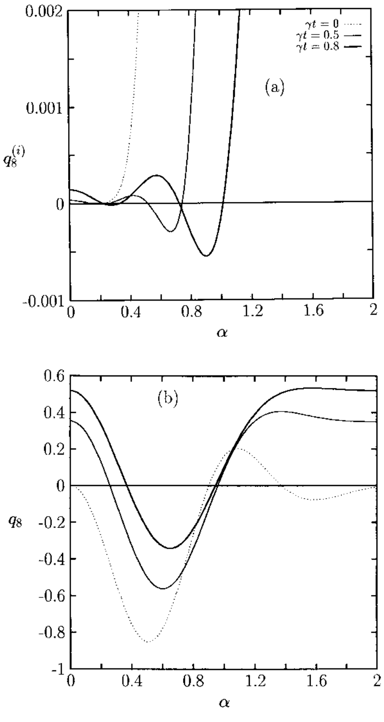


Fig. 1. Intrinsic (a) and ordinary (b) 8th-order squeezing for an ECS coupled to a thermal environment having $\bar{n}_{\mathrm{R}} = 0.1$ as function of the coherent amplitude at several times $t < t_{\mathrm{s}}$.

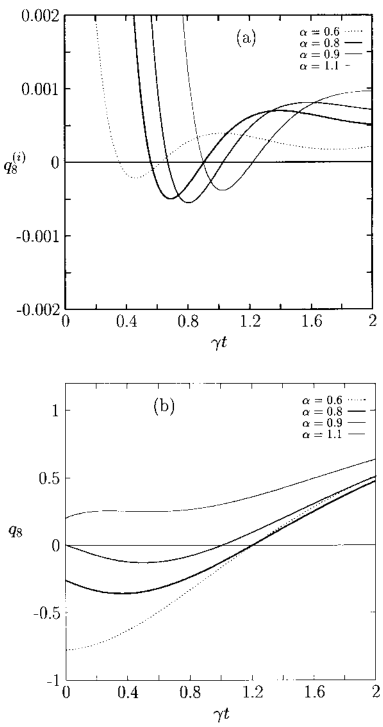


Fig. 2. Time development of intrinsic (a) and ordinary (b) 8th-order squeezing for several significant values of α selected from Figure 1.

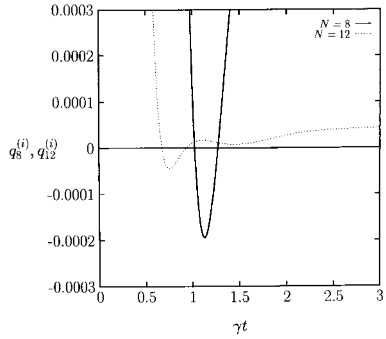


Fig. 3. Intrinsic 8th- and 12th-order squeezing *versus* γt for $\alpha = 1.2$ and $\bar{n}_R = 0.1$.

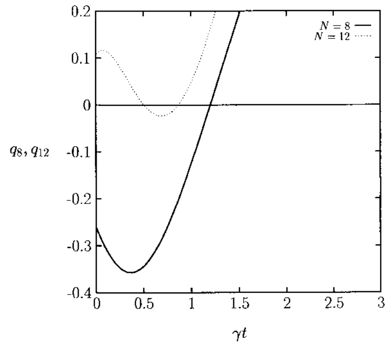


Fig. 4. Ordinary 4th- and 8th-order squeezing *versus* γt for $\alpha = 0.8$ and $\bar{n}_R = 0.1$.

We have also studied how the degree of squeezing depends on the order N. In Figure 3 we plot $q_8^{(i)}$ and $q_{12}^{(i)}$ versus time for an ECS with $\alpha = 1.2$. The maximum of the Nth-order intrinsic squeezing decreases drastically with N and is reached at N-dependent times: it moves towards smaller times when N increases. On the contrary, the maximum of the ordinary Nth-order squeezing moves towards greater times with increasing N, as can be seen in Figure 4. Analytic solutions for the times corresponding to the maximum of squeezing cannot be found because of the complexity of equations (3.9, 3.10) for higher orders.

4 Mixing by damping

4.1 2-entropy

Evaluation of the degree of mixing of the field state during the mode-reservoir interaction provides further insight to the influence that the quantum nature of the initial state has on its evolution ruled by the master equation (2.1).

To this end, we recall the quantum-mechanical counterpart of a Rényi $\tilde{\alpha}$-entropy [23],

$$S_{\tilde{\alpha}}(\rho) := \frac{1}{1-\tilde{\alpha}} \ln[\mathrm{Tr}(\rho^{\tilde{\alpha}})], \quad (\tilde{\alpha} > 0). \tag{4.1}$$

The $\tilde{\alpha} = 1$ limit of equation (4.1) is the von Neumann entropy,

$$S_1(\rho) = -\mathrm{Tr}(\rho \ln \rho), \tag{4.2}$$

which increases in a mixing process, is additive and concave [23]. An $\tilde{\alpha}$-entropy with $\tilde{\alpha} > 1$ displays the first two features, but lacks the essential property of concavity.

For a mode of radiation field, the degree of purity $\mathrm{Tr}(\rho^2)$ can be evaluated as

$$\mathrm{Tr}\{[\rho(t)]^2\} = \frac{1}{\pi} \int \mathrm{d}^2\lambda |\chi(\lambda, t)|^2 \tag{4.3}$$

(see Ref. [7] for details). It is then convenient to employ as a measure of the degree of mixing either the so-called *linear entropy*,

$$S_{\mathrm{lin}}(\rho) := 1 - \mathrm{Tr}(\rho^2), \tag{4.4}$$

invoked in references [1,2], or the special case $\tilde{\alpha} = 2$ of equation (4.1),

$$S_2(\rho) = -\ln[\mathrm{Tr}(\rho^2)]. \tag{4.5}$$

Instead of the linear entropy (4.4), which is concave, but not additive, we prefer to exploit the 2-entropy (4.5), which is a strictly monotonic function of the former,

$$S_2(\rho) = -\ln[1 - S_{\mathrm{lin}}(\rho)]. \tag{4.6}$$

4.2 ECS input

When using the CF (2.9) *via* equation (3.2) for a thermalized ECS, the integral (4.3) can be easily performed. (See Ref. [22], Appendix A, Eqs. (A6, A8).) We have derived the exact formula

$$S_2[(\rho(t)] = \ln\{2[2\bar{n}_T(t) + 1]\} - \ln\left\{1 + \left[\frac{\cosh(|\alpha|^2\eta(t))}{\cosh(|\alpha|^2)}\right]^2\right\}, \tag{4.7}$$

where the dimensionless parameter

$$\eta(t) := 1 - \frac{2\exp(-\gamma t)}{2\bar{n}_T(t) + 1} \tag{4.8}$$

is determined by the reservoir only. $\eta(t)$ is a strictly increasing and concave function of time, which varies from $\eta(0) = -1$ to $\eta(\infty) = 1$, having a single zero, namely,

$$\eta(t_m) = 0 \quad \text{for} \quad t_m = \frac{1}{\gamma} \ln\left(1 + \frac{1}{2\bar{n}_R + 1}\right) \cdot \tag{4.9}$$

Let us note the initial and equilibrium values of the 2-entropy (4.7),

$$S_2[\rho(0)] = 0, \quad S_2[\rho(\infty)] = \ln(2\bar{n}_R + 1), \tag{4.10}$$

as well as its first time derivative,

$$\frac{\partial S_2}{\partial t} = \gamma u(\eta), \tag{4.11}$$

$$\frac{\partial u(\eta)}{\partial \eta} = -\frac{1}{\bar{n}_{\rm R}} v(\eta) \left\{ v(\eta) + \frac{|\alpha|^2 \sinh(2|\alpha|^2\eta)}{[\cosh(|\alpha|^2)]^2 + [\cosh(|\alpha|^2\eta)]^2} \right\} - \frac{2(1-\eta)[1+\bar{n}_{\rm R}(1-\eta)]|\alpha|^4}{[\cosh(|\alpha|^2)]^2 + [\cosh(|\alpha|^2\eta)]^2}$$
$$\times \left\{ 1 + 2[\sinh(|\alpha|^2\eta)]^2 \frac{[\cosh(|\alpha|^2)]^2 - [\cosh(|\alpha|^2\eta)]^2}{[\cosh(|\alpha|^2)]^2 + [\cosh(|\alpha|^2\eta)]^2} \right\}. \quad (4.17)$$

where

$$u(\eta) := (1-\eta)v(\eta), \quad (4.12)$$

and

$$v(\eta) := \bar{n}_{\rm R} - [1+\bar{n}_{\rm R}(1-\eta)] \times \frac{|\alpha|^2 \sinh(2|\alpha|^2\eta)}{[\cosh(|\alpha|^2)]^2 + [\cosh(|\alpha|^2\eta)]^2} \cdot \quad (4.13)$$

Obviously, $u(0) = v(0) = \bar{n}_{\rm R}$. Making use of the function (4.13) and its derivative,

$$\frac{\partial v(\eta)}{\partial \eta} = \frac{|\alpha|^2}{[\cosh(|\alpha|^2)]^2 + [\cosh(|\alpha|^2\eta)]^2} \times \left\{ \bar{n}_{\rm R} \sinh(2|\alpha|^2\eta) - 2[1+\bar{n}_{\rm R}(1-\eta)]|\alpha|^2 \times \left[1 + \frac{2[\cosh(|\alpha|^2)]^2[\sinh(|\alpha|^2\eta)]^2}{[\cosh(|\alpha|^2)]^2 + [\cosh(|\alpha|^2\eta)]^2} \right] \right\}, \quad (4.14)$$

we evaluate the derivative of equation (4.11),

$$\frac{\partial^2 S_2}{\partial t^2} = \gamma \dot{\eta} \frac{\partial u}{\partial \eta}, \quad (4.15)$$

via the general formula

$$\frac{\partial u(\eta)}{\partial \eta} = -v(\eta) + (1-\eta)\frac{\partial v(\eta)}{\partial \eta} \cdot \quad (4.16)$$

For $\bar{n}_{\rm R} > 0$, we find

see equation (4.17) above.

In the time interval $[0, t_{\rm m}]$, the function $v(\eta)$ is positive and strictly decreasing,

$$v(\eta) \geq \bar{n}_{\rm R}, \quad \frac{\partial v(\eta)}{\partial \eta} < 0, \quad \text{for} \quad \eta \leq 0. \quad (4.18)$$

Accordingly, the 2-entropy increases significantly from zero to the value

$$S_2[\rho(t_{\rm m})] = \ln\left(2\frac{2\bar{n}_{\rm R}+1}{\bar{n}_{\rm R}+1}\right) - \ln\{1 + [\mathrm{sech}(|\alpha|^2)]^2\}, \quad (4.19)$$

which provides a measure of the degree of mixing. This is a reason for calling $t_{\rm m}$ the *mixing time*. We also define the initial value of the slope (4.11) as a conventional *rate of mixing*,

$$\Gamma_{\rm m} := \frac{\partial S_2}{\partial t}\Big|_{t=0}, \quad (4.20)$$

and get

$$\Gamma_{\rm m} = 2\gamma[\bar{n}_{\rm R} + (2\bar{n}_{\rm R}+1)\bar{n}(0)]. \quad (4.21)$$

Recall that $\bar{n}(0)$, equation (3.4), is the expectation value of the photon number in the initial ECS.

In contrast to the mixing time $t_{\rm m}$, equation (4.9), that does not depend on $|\alpha|^2$, the rate of mixing (4.21) increases with $|\alpha|^2$. Remark that the slope (4.11) decreases from $\Gamma_{\rm m}$ to a value independent of $|\alpha|^2$,

$$\frac{\partial S_2}{\partial t}\Big|_{t=t_{\rm m}} = \gamma \bar{n}_{\rm R}. \quad (4.22)$$

In order to investigate the behaviour of the 2-entropy (4.7) for $t > t_{\rm m}$, we need the asymptotic values of the functions (4.13, 4.14, 4.16), *i.e.*,

$$v(1) = \bar{n}_{\rm R} - \bar{n}(0), \quad (4.23)$$

$$\frac{\partial v(\eta)}{\partial \eta}\Big|_{\eta=1} = \bar{n}(0)[\bar{n}_{\rm R} - |\alpha|^2 \coth(|\alpha|^2)], \quad (4.24)$$

$$\frac{\partial u(\eta)}{\partial \eta}\Big|_{\eta=1} = -v(1). \quad (4.25)$$

We first indicate the role played by the sign of the derivative (4.24).

Let us assume that this quantity is positive:

$$\frac{\partial v(\eta)}{\partial \eta}\Big|_{\eta=1} > 0 \Longleftrightarrow |\alpha|^2 \coth(|\alpha|^2) < \bar{n}_{\rm R}. \quad (4.26)$$

Then, the opposite signs of $\partial v(\eta)/\partial\eta$ in equations (4.18, 4.26) prove the existence of a minimum of the function $v(\eta)$ at a value $\eta_1 \in (0,1)$ which satisfies the transcendental equation $\partial v(\eta)/\partial\eta\big|_{\eta=\eta_1} = 0$. It is easy to find a positive lower bound for this minimum:

$$v(\eta_1) > \bar{n}_{\rm R}[\mathrm{sech}(|\alpha|^2\eta_1)]^2. \quad (4.27)$$

Therefore, the condition (4.26) entails the positivity of the function $v(\eta)$, equation (4.13):

$$v(\eta) > 0, \quad \eta \in [-1, 1]. \quad (4.28)$$

Let us suppose now that, on the contrary, the derivative (4.24) is non-positive:

$$\frac{\partial v(\eta)}{\partial \eta}\Big|_{\eta=1} \leq 0 \Longleftrightarrow |\alpha|^2 \coth(|\alpha|^2) \geq \bar{n}_{\mathrm{R}}. \qquad (4.29)$$

This condition leads to the inequality

$$\frac{\partial v(\eta)}{\partial \eta} \leq -\frac{2|\alpha|^4 \cosh(|\alpha|^2\eta)}{[\cosh(|\alpha|^2)]^2 + [\cosh(|\alpha|^2\eta)]^2} \times \frac{\sinh[|\alpha|^2(1-\eta)]}{\sinh(|\alpha|^2)}, \qquad (4.30)$$

valid for $\eta \in [0,1]$. In conjunction with equation (4.18), it gives

$$\frac{\partial v(\eta)}{\partial \eta} < 0 \quad \text{for} \quad \eta \in [-1,1). \qquad (4.31)$$

Equation (4.31) shows that $v(\eta)$ is a strictly decreasing function provided that the condition (4.29) is observed.

Second, a crucial point in our analysis that has to be considered jointly with the above conclusions is the sign of the difference (4.23).

4.3 Classical regime

The inequality

$$v(1) \geq 0 \Longleftrightarrow \bar{n}(0) \leq \bar{n}_{\mathrm{R}} \qquad (4.32)$$

is compatible with equation (4.26), as well as with equation (4.29). In the first situation, the inequality (4.28) holds, while in the second, the function $v(\eta)$ steadily decreases from the positive value $v(-1)$ to the nonnegative one (4.23). Therefore, in both cases the function $v(\eta)$ is positive:

$$v(\eta) > 0, \quad \eta \in [-1,1). \qquad (4.33)$$

On the one hand, equations (4.16, 4.18) for $\eta \leq 0$, and on the other hand, equations (4.17, 4.33) for $\eta > 0$ yield the inequality

$$\frac{\partial u(\eta)}{\partial \eta} < 0, \quad \eta \in [-1,1). \qquad (4.34)$$

Accordingly, the time derivatives (4.11, 4.15) have definite signs,

$$\frac{\partial S_2}{\partial t} > 0, \qquad \frac{\partial^2 S_2}{\partial t^2} < 0, \qquad (4.35)$$

so that the 2-entropy $S_2[\rho(t)]$ (Eq. (4.7)) is a strictly *increasing* and *concave* function of time. In view of equation (4.32), this happens when the mean occupancy of the mode increases on account of the reservoir or at least is left constant. Nevertheless, as the initial ECS has a zero-mean mode amplitude, $\bar{a}(0) := \langle a \rangle_0 = 0$, it follows that, in our case,

$$\bar{n}(0) = \langle (a^\dagger - (\langle a \rangle_0)^*)(a - \langle a \rangle_0) \rangle_0. \qquad (4.36)$$

Consequently, we may cast the condition (4.32) into a more general form:

$$\bar{n}(0) - |\bar{a}(0)|^2 \leq \bar{n}_{\mathrm{R}}. \qquad (4.37)$$

When the last condition is fulfilled by an input Gaussian state, we have found in reference [7] a similar monotonic evolution of the 2-entropy for the corresponding damped mode. In particular, this is true for an initial coherent state regardless of its mean photon number. Indeed, in that case, the l.h.s of the inequality (4.37) vanishes. It seems thus appropriate to term as *classical* this regime of mixing.

4.4 Nonclassical regime

In the opposite case,

$$v(1) < 0 \Longleftrightarrow \bar{n}(0) > \bar{n}_{\mathrm{R}}, \qquad (4.38)$$

an average transfer of photons occurs from the mode to the thermal bath during their contact. Being stronger than the condition (4.29), the inequality (4.38) has also the outcome (4.31). Therefore, in the interval $[0,1]$, the function $v(\eta)$ (Eq. (4.13)), strictly decreases from the nonnegative value $v(0) = \bar{n}_{\mathrm{R}}$ to the negative one (4.23). It follows that it has a single nonnegative zero, $\eta_{\mathrm{M}} := \eta(t_{\mathrm{M}})$, with $\eta_{\mathrm{M}} \geq 0$, $t_{\mathrm{M}} \geq t_{\mathrm{m}}$. By definition, $v(\eta_{\mathrm{M}}) = u(\eta_{\mathrm{M}}) = 0$. According to equation (4.11), at the time t_{M} which depends on $|\alpha|^2$, the 2-entropy (4.7) has a unique maximum. After reaching it, the function $S_2[\rho(t)]$ steadily diminishes to its equilibrium value (4.10): this decrease for $t \geq t_{\mathrm{M}}$ reflects a *demixing* imposed by the reservoir. The property

$$v(\eta) > 0 \ \text{ for } \ \eta < \eta_{\mathrm{M}} \qquad (4.39)$$

implies *via* equations (4.16, 4.31) the inequality

$$\frac{\partial u}{\partial \eta} < 0 \ \text{ for } \ \eta \leq \eta_{\mathrm{M}}. \qquad (4.40)$$

On account of equation (4.15),

$$\frac{\partial^2 S}{\partial t^2} < 0 \ \text{ for } \ t \leq t_{\mathrm{M}}. \qquad (4.41)$$

In this way, for $t \leq t_{\mathrm{M}}$, the 2-entropy is an increasing and concave function of time. However, in the case (4.38), according to equations (4.15, 4.25), it is asymptotically convex:

$$\lim_{t \to \infty} \frac{\partial^2 S}{\partial t^2} > 0. \qquad (4.42)$$

This ensures that the 2-entropy (4.7) has a unique inflection point at $t_{\mathrm{I}} > t_{\mathrm{M}}$ with $\eta_{\mathrm{I}} := \eta(t_{\mathrm{I}}) > \eta_{\mathrm{M}}$ satisfying the transcendental equation $\partial u/\partial \eta\big|_{\eta=\eta_{\mathrm{I}}} = 0$.

$$\frac{\partial S}{\partial(|\alpha|^2)} = \frac{2\cosh(|\alpha|^2\eta)\sinh[|\alpha|^2(1-|\eta|)] + (1-|\eta|)\cosh(|\alpha|^2)\sinh(2|\alpha|^2|\eta|)}{\cosh(|\alpha|^2)\left\{[\cosh(|\alpha|^2)]^2 + [\cosh(|\alpha|^2\eta)]^2\right\}}, \tag{4.45}$$

The existence of a maximum of the 2-entropy, depending on the initial nonclassical state, is undoubtedly a quantum effect emphasized also for Gaussian states [7]. Of course, this regime of mixing deserves the attribute *nonclassical* employed here.

4.5 Dissipation

We designate as dissipation the damping of a field in contact with a zero-temperature reservoir, when $\bar{n}_R = 0$. In this limit case, the condition (4.38) is always fulfilled. In other words, the mixing by dissipation takes place in an essentially nonclassical regime. The 2-entropy reaches its maximum precisely at the mixing time

$$t_M = t_m = \frac{1}{\gamma}\ln 2, \quad \eta_M = 0, \tag{4.43}$$

so that equation (4.19) specializes to

$$S_2[\rho(t_M)] = -\ln\{\frac{1}{2}[1 + (\mathrm{sech}(|\alpha|^2))^2]\}. \tag{4.44}$$

The dissipation of the field mode specifically includes a mixing of the initial ECS with the rate $\Gamma_m = 2\gamma\bar{n}(0)$, followed at times $t > t_M$ by a *complete* demixing ending in the vacuum state.

Note also that during dissipation, according to equation (3.8), the P-representation of the density operator *does not exist at any time* and squeezing to arbitrary order is always manifest, as shown by equation (2.12).

4.6 $|\alpha|^2$-Enhancement of mixing

The derivative of the 2-entropy (4.7) with respect to $|\alpha|^2$,

see equation (4.45) above

is nonnegative:

$$\frac{\partial S}{\partial(|\alpha|^2)} > 0 \quad \text{for} \quad |\eta| < 1,$$
$$\frac{\partial S}{\partial(|\alpha|^2)} = 0 \quad \text{for} \quad |\eta| = 1. \tag{4.46}$$

This means that, except for the initial and asymptotic values (4.10), which are independent of $|\alpha|^2$, all other values of the 2-entropy increase with $|\alpha|^2$ from the lower bound

$$\lim_{|\alpha|\to 0} S_2[\rho(t)] = \ln[2\bar{n}_T(t) + 1] \tag{4.47}$$

to the upper one

$$\lim_{|\alpha|\to\infty} S_2[\rho(t)] = \ln\{2[2\bar{n}_T(t) + 1]\}. \tag{4.48}$$

This enhancement of mixing in the course of the field-reservoir interaction by increasing the squared modulus of the coherent amplitude α is a quantum effect of the initial state on the mode attenuation. When $|\alpha|^2$ is large enough to satisfy the condition (4.38), the 2-entropy exhibits a maximum at a finite time t_M greater than t_m, equation (4.9). We record a substantial loss of information about the properties of the field state up to the time t_m, which is less than the nonclassicality time t_c, equation (3.8).

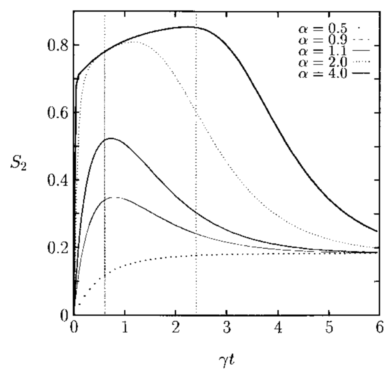
What happens in fact when $|\alpha|^2$ becomes larger? With increasing $|\alpha|^2$, the overlap of the two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ diminishes, so that their initial superposition becomes more and more conspicuous. Owing to the environment, the *Schrödinger cat* (3.1) appears to be more and more fragile, as the rate of mixing (4.21) also reveals. Equation (4.46) shows that the damped mode is more able to deviate from the initial state by mixing to a higher degree.

Notice finally that the properties proved by analytic means in this section are illustrated in Figure 5 as follows. In Figure 5a we have plotted 2-entropy *versus* γt for several values of α when $\bar{n}_R = 0.1$, ($\gamma t_m = 0.606$, $\gamma t_s = 1.792$, $\gamma t_c = 2.397$). The maximum of the 2-entropy is present only if α satisfies the condition (4.38). The value $\alpha = 0.5$ is in the classical range, while all other values of α belong to the nonclassical regime of mixing. The vertical line drawn at $\gamma t_m = 0.606$ illustrates the time of significant mixing, and the line $\gamma t_c = 2.397$ delimitates the time interval when the state of the damped mode is still nonclassical. Figure 5b is a plot of the transient 2-entropy in the limit case of a zero-temperature reservoir. The damped mode is nonclassical at any time. The mixing regime too is always nonclassical and the maximum of the 2-entropy is reached at $\gamma t_M = \ln 2 = 0.693$ regardless of α. Remark that for higher values of α the process of mixing has a longer duration and the maximum of the 2-entropy enlarges in both situations (a) and (b).

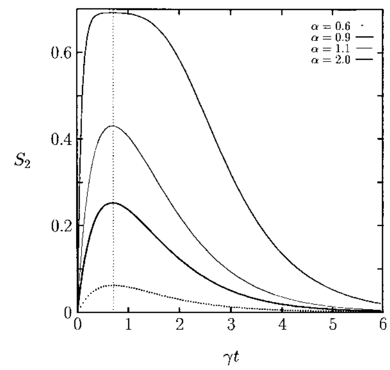
5 Conclusions

We have shown that a heat bath may allow new manifestations of the quantum nature of the field mode it interacts with. For instance, transient higher-order squeezing is possible only if the thermal mean occupancy transferred to the mode is low enough. Therefore, this effect could be of interest in experimental attempts to obtain nonclassical field states which are conducted at very low temperatures.

On the other hand, for an ECS input, we have examined the features of the mixing during the mode-reservoir interaction. A nonclassical behaviour of the mixing process is characterized by the existence of a maximum in the evolution of the 2-entropy. This occurs when the mean photon



(a)



(b)

Fig. 5. 2-entropy production governed by the master equation (2.1) for an input ECS plotted at several values of α and for $\bar{n}_{\mathrm{R}} = 0.1$ (a) and zero-temperature bath (b). Notice that in the case (b) the position of the maximum is independent of α.

number in the initial mode exceeds the mean occupancy of the bath. A similar nonclassical regime of mixing has been pointed out in reference [7] for a class of Gaussian states evolving according to the quantum optical master equation.

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